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LETTERS TO THE EDITOR

A NOTE ON THE EFFECT OF HUB INERTIA AND PAYLOAD ON THE VIBRATION OF A FLEXIBLE SLEWING LINK

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1. INTRODUCTION

Typical experimental prototypes of flexible robot arms are constituted as a rotating beam with distributed flexibility. The simplest example of flexible manipulators is a flexible slewing beam constrained to move in a horizontal plane. The beam is usually actuated by a motor and may carry a payload. It occurs in such diverse applications as a link of a flexible robot arm, a flexible antenna, a helicopter blade, and also as part of a flexible structure [1–6].

The problem of bending vibration of a rotating beam undergoing large motions has been considered for many decades. For example, a work by Lo [7] considered the non-linear term which arises from the Coriolis acceleration, while reference [8] included all inertia effects, geometrical non-linearities, and coupling of extensional-flexural deformations. Eick and Mignolet [9] investigated singular and regular perturbation methods as applied to the natural frequencies and mode shapes of a rotating beam.

The rotating beam system is often modelled as an Euler–Bernoulli beam with clamped-free or pinned-free boundary conditions. The classical model neglects the effect of any payload, and also that of the hub inertia on the system dynamics. To demonstrate the use of variational calculus for the slewing beam system, Morris and Taylor [10] provided a detailed derivation based on a clamped model studied in references [11–13].

Recent works [5, 12–16] addressed an importance issue, that of selecting different sets of modes for problems of elastic beams that undergo large rigid-body displacements. Experimental results [5, 14] suggested that the exact natural frequencies are intermediate between the clamped and pinned cases. Shabana [16] demonstrated that different mode shapes that correspond to different sets of natural frequencies can be used to obtain the same resonance conditions by using simple co-ordinate transformations. Bellezza *et al*. [12] developed a mathematical model for a flexible slewing beam, in which two formulations with clamped and pinned roots are each modelled and compared. The frequency equation of the rotating loaded-beam system was also generated.

The objective of this note is to re-tackle the issue of boundary conditions in the modelling of slewing beam systems, by taking account of those results given in some recent works. A rotating beam driven by a motor and carrying a payload is considered in this note. Analytical frequencies by virtue of the developed model are compared with those obtained experimentally.

2. MODEL CONSIDERED AND ITS FREQUENCY EQUATION

The considered model is made of an actuator at the base with total (rotor and hub) inertia J_0, a flexible beam with uniform linear mass density ρA, length l and a payload

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of mass M_p and inertia J_p. The overall structure rotates in an inertia frame (x_0, y_0) and the non-inertial frame of reference (x, y). The rotating beam is constrained on an horizontal plane, so no gravity effect is taken into account. It is assumed that the deflection is small.

As shown in Figure 1, the non-inertial co-ordinate is chosen so that the x-axis is tangent to the link at the hub. Note that $v(x, t)$ is the deflection of point P in (x, y), $\theta(t)$ the angle between (x_0, y_0) and (x, y), $\alpha(x, t)$ the angular position of point P in (x_0, y_0), and $\tau(t)$ is an applied torque at the hub.

The derivation and solution of the Bernoulli–Euler beam equation can be found in many textbooks on structural dynamics [17 and 18, for example]. Considering a differential segment of the uniform beam, the resulting equation for free vibrations is

$$(EI/l^4)(\partial v(\xi, t)/\partial \xi^4) + \rho A \partial^2 v(\xi, t)/\partial t^2 = 0, \quad (1)$$

where E is Young's modulus of elasticity, I is the cross-sectional moment of inertia, ξ $(=x/l)$ is the normalized spatial variable along the long axis of the beam, [0, 1], x is the spatial variable along the long axis of the beam, $[0, l]$, t is time, and $v(\xi, t)$ is the beam deflection.

Equation (1) can be solved by using the technique of separation of variables,

$$v(\xi, t) = \sum_{i=1}^{m} \varphi_i(\xi) q_i(t), \quad (2)$$

to yield the admissible modal shape function,

$$(\xi) = C_1 \cos \lambda\xi + C_2 \cosh \lambda\xi + C_3 \sin \lambda\xi + C_4 \sinh \lambda\xi, \quad (3)$$

where $\lambda = \beta l$ and $\beta^4 = (\omega^2 \rho A)/(EI)$.

A shape function involving the motor inertia and payload was given as [12]

$$\varphi(x) = C_1 \cos \beta x + C_2 \cosh \beta x + C_3 \sin \beta x + C_4 \sinh \beta x + Fx, \quad (4)$$

where F is a constant involving J_0, M_p and J_p. It is noted that equation (4) reduces to equation (3) if $F = 0$.

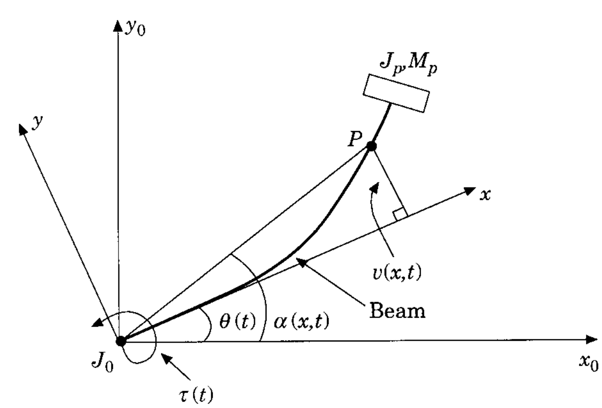

Figure 1. The flexible slewing beam considered.

TABLE 1

Eigenvalues versus α_1 ($\beta_0 = \beta_1 = 0$)

α_1	λ_1	λ_2	λ_3	λ_4	λ_5
0·0	3·9266	7·0686	10·210	13·352	16·493
0·001	3·9227	7·0616	10·200	13·339	16·477
0·01	3·8892	7·0032	10·119	13·235	16·354
0·05	3·7694	6·8196	9·8906	12·977	16·074
0·5	3·3666	6·4172	9·5196	12·640	15·768
1·0	3·2733	6·3560	9·4749	12·605	15·739
5·0	3·1721	6·2988	9·4353	12·574	15·714
10·0	3·1572	6·2911	9·4301	12·570	15·711
50·0	3·1448	6·2848	9·4258	12·567	15·709
100·0	3·1432	6·2840	9·4253	12·567	15·708
1000·0	3·1418	6·2833	9·4248	12·566	15·708
5000·0	3·1416	6·2832	9·4248	12·566	15·708
pinned-free	3·9266	7·0686	10·210	13·352	16·493
pinned-pinned	3·1416	6·2832	9·4248	12·566	15·708

By substituting the associated boundary conditions into equation (4), the eigenvalues can be obtained from the eigen-analysis [10, 12, 19] and its dimensionless form is given by

$$\begin{aligned}&\mathrm{csh} - \mathrm{sch} - 2\alpha_1\lambda\,\mathrm{ssh} - \beta_0\lambda^3(1 + \mathrm{cch}) - 2\beta_1\lambda^3\mathrm{cch} - \alpha_1\lambda^4(\beta_0 + \beta_1)(\mathrm{csh} - \mathrm{sch})\\ &\quad + \beta_0\beta_1\lambda^6(\mathrm{csh} + \mathrm{sch}) - \beta_0\beta_1\alpha_1\lambda^7(1 - \mathrm{cch}) = 0, \qquad (5)\end{aligned}$$

where $\mathrm{c} = \cos\lambda$, $\mathrm{s} = \sin\lambda$, $\mathrm{ch} = \cosh\lambda$ and $\mathrm{sh} = \sinh\lambda$, while $\beta_0 = J_0/\rho A l^3$, $\alpha_1 = M_p/\rho A l$ and $\beta_1 = J_p/\rho A l^3$.

3. NUMERICAL AND EXPERIMENTAL RESULTS

The effect of the payload's mass on the eigenvalues λ_i is illustrated in Table 1, while the effect of the hub's and payload's inertias is reflected in Tables 2 and 3, respectively. The frequencies due to the change of a payload with large inertia ($\beta_1 = 1000$) are given in

TABLE 2

Eigenvalues versus β_0 ($\alpha_1 = \beta_1 = 0$)

β_0	λ_1	λ_2	λ_3	λ_4	λ_5
0·0	3·9266	7·0686	10·210	13·352	16·493
0·001	3·8978	6·8763	9·5525	11·951	14·582
0·01	3·6405	5·6160	8·0841	11·075	14·174
0·05	2·9675	4·8972	7·8970	11·011	14·144
0·5	2·1201	4·7136	7·8589	10·997	14·138
1·0	2·0100	4·7038	7·8568	10·996	14·138
5·0	1·9047	4·6960	7·8552	10·996	14·137
10·0	1·8901	4·6951	7·8550	10·996	14·137
50·0	1·8781	4·6943	7·8548	10·996	14·137
100·0	1·8766	4·6942	7·8548	10·996	14·137
1000·0	1·8753	4·6941	7·8548	10·996	14·137
5000·0	1·8751	4·6941	7·8548	10·996	14·137
pinned-free	3·9266	7·0686	10·210	13·352	16·493
clamped-free	1·8751	4·6941	7·8548	10·996	14·137

TABLE 3

Eigenvalues versus β_1 ($\beta_0 = 0$ *and* $\alpha_1 = 0.01$)

β_1	λ_1	λ_2	λ_3	λ_4	λ_5
0·0	3·8892	7·0032	10·119	13·235	16·354
0·001	3·8304	6·6270	8·9895	11·448	14·300
0·01	3·3557	5·2092	7·9350	10·987	14·092
0·05	2·5501	4·7932	7·8391	10·952	14·076
0·5	1·7767	4·6996	7·8186	10·945	14·072
1·0	1·6806	4·6945	7·8175	10·944	14·072
5·0	1·5888	4·6905	7·8166	10·944	14·072
10·0	1·5761	4·6900	7·8165	10·944	14·072
50·0	1·5657	4·6896	7·8164	10·944	14·072
100·0	1·5644	4·6895	7·8164	10·944	14·072
1000·0	1·5632	4·6895	7·8164	10·944	14·072
5000·0	1·5631	4·6895	7·8164	10·944	14·072

Table 4. It is obvious that the classical cases can be deduced from these tables: pinned-free ($\alpha_1 = \beta_0 = \beta_1 = 0$), pinned-pinned ($\alpha_1 = 5000$ and $\beta_0 = \beta_1 = 0$), and clamped-free ($\beta_0 = 5000$ and $\alpha_1 = \beta_1 = 0$). Also seen in Tables 1–4 is the following: (1) Frequencies decrease as the parameters (α_1, β_0 and β_1) increase. (2) Changes on eigenvalues are significant only for lower modes, while the changes converge faster for higher modes. (3) Inertia terms (β_0 and β_1) influence more on eigenvalues if compared with the effect of linear mass (α_1). (4) The system can be quite flexible (low eigenvalue) for large tip-mass/inertia, as seen in Table 4.

Three beams of $700 \times 25{\cdot}5 \times 2\ \mathrm{mm}^3$, $845 \times 25{\cdot}5 \times 2\ \mathrm{mm}^3$ and $995 \times 25{\cdot}5 \times 2\ \mathrm{mm}^3$ were tested by using a high speed camera [5]. They were driven and rotated by motor with inertia of $4{\cdot}28 \times 10^{-3}\ \mathrm{kg\,m}^2$. The resonance frequencies at different modes for each respective beam were measured. It is demonstrated in Table 5 that the frequency equation (5) can well predict frequencies of the tested beams. As stated in references [5, 10, 14], the system would be neither pinned-free nor have clamped-free boundary condition modes. Rather, the mode shapes themselves are a function of the feedback control [14]. It has been shown in Table 2 that the actual frequency depends on the motor inertia, J_0.

TABLE 4

Eigenvalues versus α_1 ($\beta_0 = 0$ *and* $\beta_1 = 1000$)

α_1	λ_1	λ_2	λ_3	λ_4	λ_5
0·001	1·5701	4·7100	7·8501	10·990	14·130
0·01	1·5632	4·6895	7·8164	10·944	14·072
0·05	1·5339	4·6094	7·6940	10·786	13·885
0·5	1·3199	4·2372	7·2808	10·370	13·480
1·0	1·1920	4·1197	7·1901	10·298	13·421
5·0	0·86108	3·9746	7·0960	10·229	13·367
10·0	0·73323	3·9513	7·0825	10·220	13·359
50·0	0·49982	3·9317	7·0714	10·212	13·353
100·0	0·42573	3·9292	7·0700	10·211	13·353
1000·0	0·27829	3·9269	7·0687	10·210	13·352
5000·0	0·24495	3·9267	7·0686	10·210	13·352

TABLE 5

Comparison on frequencies (Hz) of tested beams

Beam length (mm)	Mode of vibration	Frequency			
		clamped-free	equation (5)	experimental	pinned-free
700	1	3·39	9·87	9·7	14·89
700	2	21·27	24·34	23·8	48·24
845	1	2·33	7·87	7·35	10·22
845	2	14·60	18·32	17·7	33·11
995	1	1·68	4·89	5·8	7·37
995	2	10·53	12·05	14·0	23·88

4. DISCUSSION ON EQUATIONS OF MOTION

The dynamic model for the rotating beam system can be written as

$$[\mathbf{M}]\{\ddot{q}_i\}+[\mathbf{K}]\{q_i\}=\{\mathbf{f}\}\tau, \tag{6}$$

where [**M**] is the mass matrix, [**K**] the stiffness matrix, and $\{\mathbf{f}\}$ the vector associated with an applied torque $\tau(t)$.

Bellezza *et al*. [12] derived the equation of motion (6) for pseudo-clamped and pseudo-pinned models. They found that, for a pseudo-pinned model, there is no coupling between the rigid and the flexible modes through the [**M**] and [**K**] matrices, while the flexible modes were directly excited by the input since the vector $\{\mathbf{f}\}$ has all its elements different from zero. The [**K**] matrix is again diagonal for the clamped-free (or pseudo-clamped) mode shapes, whereas the vector associated with τ in this case is simply $\{\mathbf{f}\}=\{1, 0, \ldots, 0\}^t$ [12, 14]. In fact, one may apply the principle of virtual work [18] to the terms associated with slope, $\varphi'(0)$. It is obvious that the vector $\{\mathbf{f}\}$ associated with flexible modes of clamped-free case vanishes as the slope $\varphi'(0)$ is zero.

Bellezza *et al*. [12] stated that the clamped and pinned models only differ by the choice of the rotating frame. The change of reference frame does not alter the characteristic frequencies of the slewing link since the physical system remains the same. A similar conclusion was drawn by Shabana [16].

Both the pseudo-clamped and the pseudo-pinned models use generalized co-ordinates that are not directly measurable. It is then convenient to use a new set of directly measurable co-ordinates, i.e., the displacements of appropriate distinct points x_i along the beam [1, 12].

A more general model involving base excitation and a payload's offset was studied by Low [19]. It was demonstrated [19, 20] that, for a given payload, an off-set influences mainly the fundamental natural frequency.

In view of the above discussion and the results shown in Tables 1–5, it is obvious that the parameters of hub and payload should be incorporated for an accurate model, regardless of the use of the pseudo-clamped or the pseudo-pinned case.

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